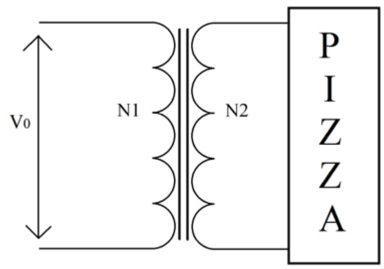
General Physics: Electromagnetism, Correction 11

Exercise 1 :

A neon signs requires a voltage of $V_2 = 12$ kV to operate. Given an input of $V_0 = 240$ V from a power line, a transformer is used to achieve the necessary voltage. What should be the ratio of the number of turns in the secondary winding to the number of turns in the primary winding of the transformer? What would be the output voltage if the transformer was connected in reverse?



Solution 1 :

An AC transformer consists of two coils of wire wound around a core of iron. The core is used to increase the magnetic flux and to provide a medium for the flux to pass from one coil to the other. For the left part of the transformer we have,

$$V_0 = -N_1 \frac{d\Phi_B}{dt}, \tag{1}$$

where Φ_B is the flux in the core and N_1 the number of turns of the left coil. For the right part of the transformer we have,

$$V_2 = -N_2 \frac{d\Phi_B}{dt}. \tag{2}$$

From the term, $d\Phi_B/dt$, we can connect V_2 to V_0 as,

$$V_0 = N_1 \frac{V_2}{N_2} \rightarrow \frac{V_2}{V_0} = \frac{N_2}{N_1} \rightarrow \frac{N_2}{N_1} = \frac{12 \cdot 10^3}{240} = 50. \tag{3}$$

If the transformer was connected the other way around, we would have a voltage which would be 50 times smaller than the input one:

$$V_2 = \frac{N_1}{N_2} \cdot V_0 = \frac{1}{50} \cdot 240 = 4.8\,\mathrm{V}. \tag{4}$$

Exercise 2 :

A coaxial cable of length l is made of a central conductive wire of radius r_1 and the surrounding tube-like outer conductor of radius r_2 ($r_1 \ll r_2 \ll l$), separated by a dielectric tube (see Figure below). These cables are widely used for transmitting high-frequency signals with the two conductors of the cable on one end connected to a source and on the other end connected to a receiver. Determine the self inductance L of such cable.

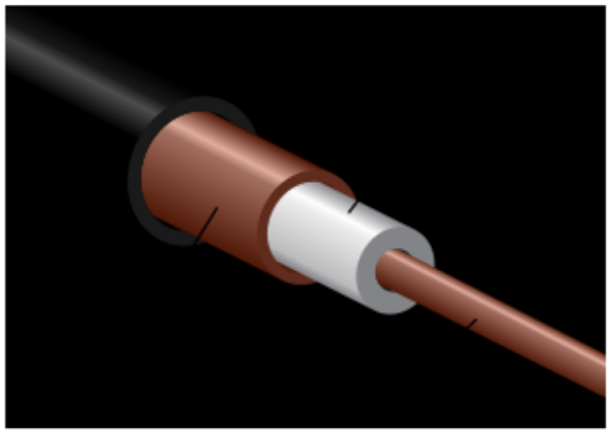


Figure 1: Coaxial cable.

Solution 2 :

This exercise can be solved in two different ways. One is using the fact that the magnetic field is only present between the two conductors and zero outside (because they both carry the same but opposite current) and using the expressions for the **energy stored in the magnetic field** of an inductor. The second way, is to calculate the **magnetic flux** through an area along the cable.

For both ways, we need to know the magnetic field first, which we can find by using Ampère's law:

$$\Phi_B = \oint \mathbf{B} \cdot d\mathbf{s} = \sum I_{\text{encl.}} B = \frac{\mu_0 I_\Sigma}{2\pi r}, \tag{5}$$

where I_Σ is the total enclosed current and r is the radius of the loop we consider. Here, we are using a constant current I, which is only correct for $r_1 < r < r_2$, but not for $r < r_1$, where the current increases with the loop size as well. For this problem, however, we can ignore this region, because $r_2 \gg r_1$, so the contribution magnetic field within the inner wire is negligible. Since we have a current I flowing in the inner conductor and an equal but opposite current flowing in the shell, $B = \frac{\mu_0 I}{2\pi r}$ between the conductors and $B = 0$ anywhere outside.

1. **Using the stored energy**
   When a current through a conductor (in this case a cable) of inductance L increases from 0 to I, energy is generated and stored in the conductor magnetic field. It relates to the inductance as follows:
   $$U = \frac{1}{2}LI^2 \tag{6}$$
   One can also express the link between the energy and the magnetic field itself:
   $$U = \int \frac{B^2}{2\mu_0} dV, \tag{7}$$
   where the expression has to be integrated over the whole volume that contains the field. Since $r_2 \gg r_1$, and we are only interested in the energy stored in the magnetic field, we can neglect the contribution of the magnetic field inside the inner wire, because its volume is much smaller than that between the wire and the shell. Therefore, we end up with:
   $$U = \int_{r_1}^{r_2} \frac{B^2(r)}{2\mu_0} dV = \int_{r_1}^{r_2} \frac{B^2(r)}{2\mu_0} 2\pi r l dr = \int_{r_1}^{r_2} \frac{\mu_0^2 I^2}{4\pi^2 r^2 2\mu_0} 2\pi r l dr = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{r_2}{r_1} \tag{8}$$
   Then, setting the two initial equations equal:
   $$\frac{1}{2}LI^2 = \frac{\mu_0 I^2 l}{4\pi} \ln \frac{r_2}{r_1} \tag{9}$$
   Therefore, by isolating the inductance, we get:
   $$L = \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1} \tag{10}$$

2. **Calculating the flux through a loop formed by the two conductors**
   In the lecture, we have defined the self inductance L starting from the electromotive force ϵ which we is also related to Faraday's law:
   $$\epsilon = -L\frac{dI}{dt} = -\frac{d\phi_B}{dt} \tag{11}$$
   In most common cases, the magnetic field is directly proportional to the current. In all these cases we can easily rewrite:
   $$L = \phi_B / I. \tag{12}$$
   For the second method, we consider a loop formed by connecting the two conductors at their ends. Because $l \gg r_2$, we can neglect the inductance of these additional connections. We end up with a cut along the cable (see brown shaded region in Figure 2), through which a current of I is flowing. At the same time, we have a magnetic flux going through the loop area.

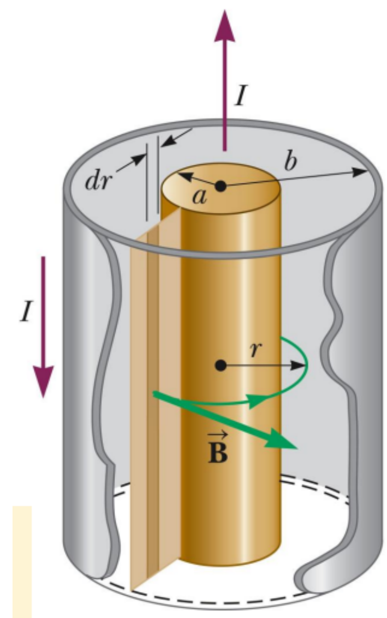


Figure 2: Coaxial cable section

To get the magnetic flux through the loop, we need to integrate the magnetic field over the loop area A, because is not homogeneous:

$$\Phi_B = \int_A \mathbf{B} \cdot d\mathbf{S} = \int_{r_1}^{r_2} Bl \cdot dr = \int_{r_1}^{r_2} \frac{\mu_0 Il}{2\pi r} \cdot dr = \frac{\mu_0 Il}{2\pi} \ln \frac{r_1}{r_2}, \tag{13}$$

where we use $dS = l \cdot dr$ for the infinitesimal area of a short stripe along the whole wire (see dark shaded region in Figure 2) and the expression for B comes from Eq. (5). Also here, we neglect the contribution of the magnetic field in the inner wire (because $r_2 \gg r_1$), by integrating only from r_1 to r_2. If we would not make this assumption, we would have to find the correct expression for B for that region, as mentioned in the beginning of this solution.

Finally, using Eq. (12), we get the same result as with the first method:

$$L = \frac{\mu_0 l}{2\pi} \ln \frac{r_2}{r_1} \tag{14}$$

Exercise 3 :

Consider a long cable that consists of two parallel straight round wires of length l, diameter $2a$, made of non-magnetic conductive material (see Figure below). The centers of the wires in the cables are separated by distance D, such that: $l \gg D \gg 2a$. When such a cable is used, the directions of the current in the wires are opposite. Derive an expression for self inductance L of the cable.

Hint: To compute the flux, consider a closed loop obtained by connecting with small conductive wires both the extremities of the two different cables. Since these two extra wires are small, you can assume they do not change the self inductance of the two wires. Remember that for calculating the self inductance of a circuit, you have to consider the magnetic flux through the circuit generated by the current flowing in the circuit. Since $D \gg a$, you can neglect the contribution of the magnetic flux through the wires themselves.

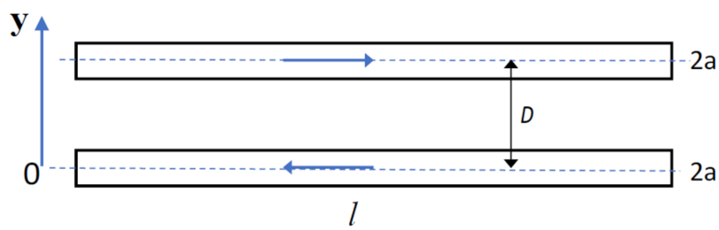


Figure 3: Two wires at distance D.

Solution 3 :

In the lecture, we have defined the self inductance L starting from the electromotive force ϵ which we is also related to Faraday's law:

$$\epsilon = -L\frac{dI}{dt} = -\frac{d\phi_B}{dt} \tag{15}$$

In most common cases, the magnetic field is directly proportional to the current. In all these cases we can easily rewrite:

$$L = \phi_B/I. \tag{16}$$

In this exercise, the wires generate a magnetic field which create circles around the wires with decreasing intensity going further from the wire (distance = r), according to the law:

$$B(r) = \frac{\mu_0 I}{2\pi r} \tag{17}$$

which has been computed by applying the Ampere's Law. In this case we consider only the field outside of the wire, using the fact that $D \gg 2a$.

In order to define the flux of a magnetic field, we need to define a closed loop, or an area over which the magnetic field goes through. We use the hint of the exercises and our closed area will be the one in between the two cables. In this region the magnetic fields generated by the two wires sum up, directed perpendicularly and towards the area (or the sheet).

As the magnetic field changes over the area we are considering, we cannot just say that $\phi_B = BA\cos(\theta)$ but we need to integrate over the surface:

$$\phi_B = \int_S \vec{B}d\vec{S} = \int_a^{D-a} (B_1(y) + B_2(y))l dy, \tag{18}$$

where $B_1(y) = \frac{\mu_0 I}{2\pi y}$ and $B_2(y) = \frac{\mu_0 I}{2\pi(D-y)}$ is the magnetic field generated by the bottom and top wire respectively. We integrate over dy as the length of the cable l is constant and the magnetic field is changing over the y direction in the picture. Plugging in B_1 and B_2, we get:

$$\phi_B = \frac{\mu_0 I l}{2\pi}\int_a^{D-a} dy\left(\frac{1}{y}+\frac{1}{D-y}\right) = \frac{\mu_0 I l}{2\pi}[2\ln(D-a) - 2\ln(a)]$$
$$\phi_B = \frac{\mu_0 I l}{\pi}\ln\left(\frac{D-a}{a}\right) \approx \frac{\mu_0 I l}{\pi}\ln\frac{D}{a}, \tag{19}$$

where in the last step we have used the fact that $D \gg a$.

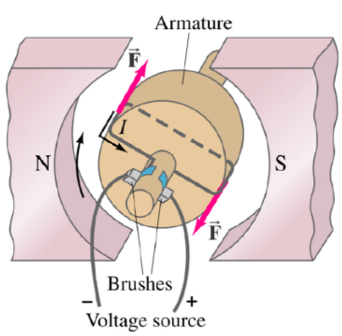
Using Eq. (16), we end up with:

$$L = \frac{\mu_0 l}{\pi}\ln\frac{D}{a} \tag{20}$$

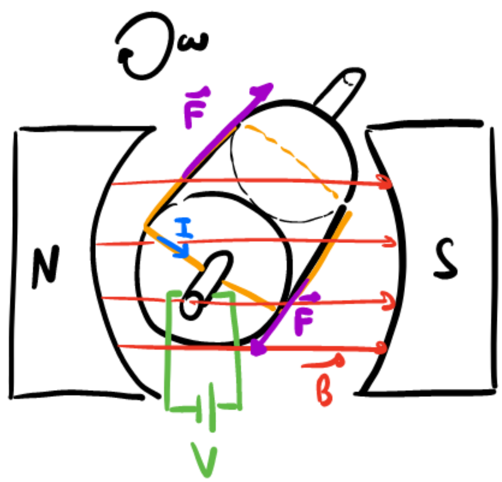
Exercise 4 :

A small electric car experiences a frictional force of 250 N when moving at a constant speed of 35 km/h. The electric motor is powered by ten batteries of 12 V each, connected in series. It contains a rectangular coil with 270 turns and of size 12×15 cm, which rotates in a magnetic field $B = 0.6$ T. The motor is directly connected (without gearbox) to the wheels with a diameter of 58 cm, so the coil and the wheels turn at the same frequency. We consider the moment when the coil is in the same plane as the magnetic field $\vec{B}$, such that the generated torque is maximal.

1. What is the current passing through the motor?
2. What is the emf induced in the coil?
3. What is the dissipated power in the coil?
4. What percentage of power produced by the motor is delivered to the wheels?

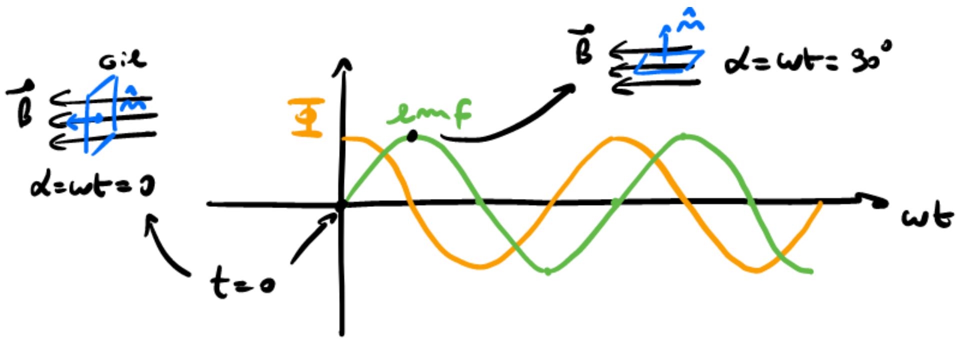


Solution 4 :



- The coil experiences a Lorentz force due to the current that flows through it and the static external magnetic field $\vec{B}$. Generally speaking, a force acting on a rotating object generates a torque τ according to:
$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{21}$$
The torque generated by the Lorentz force can be written as:
$$\vec{\tau}_B = \vec{\mu} \times \vec{B}, \tag{22}$$
where μ is the magnetic moment of the coil, $\mu = NIA_{\text{coil}}$.
Since we consider the moment when the coil is in the same plane as the magnetic field B, such that the generated torque is maximal, we have that $\vec{\mu}$ and $\vec{B}$ are perpendicular. The magnitude of the torque is then simply $|\vec{\tau}_B| = \mu B$.
In addition to the Lorentz force, we have the frictional force F_{fr} that opposes the rotation of the wheel. The torque associated to the frictional force is $\vec{\tau}_{\text{fr}} = \vec{r} \times \vec{F}_{\text{fr}}$. Its magnitude is $|\vec{\tau}_{\text{fr}}| = F_{\text{fr}} d_{\text{wheel}}/2$, being d_{wheel} the diameter of the wheel.
Since the rotation speed is constant, the two torques must equilibrate, so that $|\vec{\tau}_{\text{fr}}| = |\vec{\tau}_B|$. From this equation we can deduce the current passing through the motor
$$I = \frac{F_{\text{fr}} d_{\text{wheel}}}{2NA_{\text{coil}}B} = 24.9\ A. \tag{23}$$

- To compute the *emf* induced in the coil we use Faraday's law. Magnetic flux Φ_B is given by $\vec{B} \cdot \vec{A}_{\text{coil}}$ (we discarded the surface integral in the definition of magnetic flux since $\vec{B}$ is uniform). Therefore, calling ω the rotation speed, we have $\Phi_B = BA_{\text{coil}} \cos(\omega t)$. The *emf* then reads
$$emf = -\frac{\partial \Phi_B}{\partial t} = BA_{\text{coil}}\omega \sin(\omega t). \tag{24}$$
We notice that $\Phi_B \propto \cos(\omega t)$ while $emf \propto \sin(\omega t)$, so flux and *emf* are out-of-phase of 90 degrees.



- At t=0, Φ_B is maximal and the emf is zero.

- At $t = \frac{\pi}{2}$, Φ_B is zero and the emf is maximal:

$$emf_{\text{max}} = BNA_{\text{coil}}\omega = BNA_{coil}\frac{v}{d_{\text{wheel}}/2} = 2BNA_{\text{coil}}\frac{v}{d_{\text{wheel}}} = 97.8\ V, \tag{25}$$

where the fact that $v = \omega r$ has been used.

- To compute the dissipated power in the coil, we just use the definition of power as $P = IV$. The generated power is given by

$$P_{\text{gen}} = V_{\text{batteries}} \cdot I = 10 \cdot 12 \cdot 24.9 = 2.988\ kW, \tag{26}$$

while the power due to the induced *emf*

$$P_{\text{ind}} = emf \cdot I = 97.8 \cdot 224.9 = 2.435\ kW. \tag{27}$$

Finally, the dissipated power reads

$$P_{\text{diss}} = P_{\text{gen}} - P_{\text{ind}} = 553\ W. \tag{28}$$

- The percentage of power produced by the motor and delivered to the wheels is given by

$$\eta = \frac{P_{\text{ind}}}{P_{\text{gen}}} = \frac{2.435}{2.988} = 81\%. \tag{29}$$

Exercise 5 :

A square conductive loop with side length $l = 10\,\text{cm}$ and total resistance $R = 10\,\Omega$ is immersed in a uniform magnetic field $B = 0.52\,\text{T}$. The loop is constrained to rotate about a fixed axis orthogonal to the direction of the field, as shown in the figure below. Let θ be the angle between the direction of the magnetic field and the normal to the loop. The loop is kept in rotation with a constant angular velocity $\omega = 12.4\,\text{rad/s}$.

Determine:

(a) the absolute value of the electric charge passing through the loop during half a turn between the positions $\theta = 0$ and $\theta = \pi$;

(b) the external mechanical torque required to maintain the loop in rotation, and its average value over one turn;

(c) the work done by this mechanical torque over one turn. Remember that the work done by a torque τ is given by $W = \int_{\theta_1}^{\theta_2} \tau d\theta$, where θ_1 and θ_2 represents the initial and final angular positions.

(d) the energy dissipated in the loop over one turn.

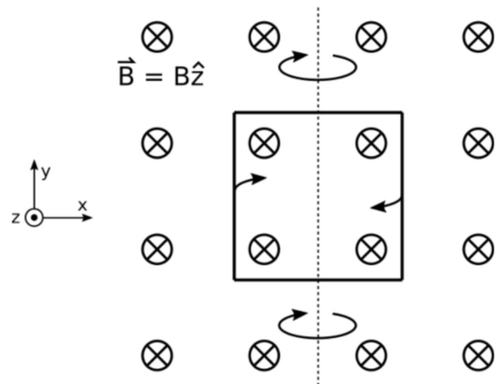


Figure 4: Square conducting loop in magnetic field

Solution 5 :

(a)

The induced electromotive force (emf) due to the magnetic field acts on the rotating loop. The current circulating in the loop varies with time and is given by Faraday-Neumann-Lenz's law. Assuming that at time $t = 0$ the loop is in the position $\theta = 0$, we have:

$$I(t) = \frac{1}{R}\frac{d\Phi_B}{dt} = -\frac{1}{R}\frac{d}{dt}\left(Bl^2 \cos\theta\right) = \frac{Bl^2\omega}{R}\sin\omega t.$$

This current, during the first half-turn, flows counterclockwise when viewed from the side of the loop's normal. To determine the value of the charge $Q_{1/2}$ passing through the loop during the first half-turn, we integrate the current over time from $t = 0$ (corresponding to $\theta = 0$) to $t = \pi/\omega$ (corresponding to $\theta = \pi$):

$$Q_{1/2} = \int_0^{\pi/\omega} I(t)\,dt = \frac{Bl^2\omega}{R}\int_0^{\pi/\omega} \sin\omega t\,dt = \frac{2Bl^2}{R} = 1.04\,\mathrm{mC}.$$

(b)

To maintain the motion at a constant angular velocity, an external mechanical torque is needed to counteract the resistive torque due to the magnetic forces acting on the sides of the loop. This resistive torque is always directed opposite to the direction of the angular velocity vector ω.

We calculate the magnitude of the resistive torque by equating the resistive moment to the product of the forces acting on the vertical sides of the loop and multiplying by the lever arm:

$$|M(t)| = |M_B(t)| = |I(t)Bl^2 \sin \omega t| = \frac{B^2 l^4 \omega}{R} \sin^2 \omega t.$$

As seen, this value is always positive. To determine the average torque over one turn, we integrate the function $M(t)$ over $t = 0$ to $t = T = 2\pi/\omega$ and divide by the period T. Thus:

$$\overline{M} = \frac{1}{T} \int_0^T M(t)\, dt = \frac{B^2 l^4 \omega}{2\pi R} \int_0^{2\pi/\omega} \sin^2 \omega t \, dt = \frac{B^2 l^4 \omega}{2R} = 1.7 \times 10^{-5}\,\text{Nm}.$$

(c)

The work done by the external torque over one turn can be obtained by expressing the torque as a function of the angle θ and integrating over θ from $\theta = 0$ to $\theta = 2\pi$:

$$L_M = \int_0^{2\pi} M(\theta)\, d\theta = \frac{B^2 l^4 \omega}{R} \int_0^{2\pi} \sin^2 \theta \, d\theta = \frac{\pi B^2 l^4 \omega}{R} = 1.05 \times 10^{-3}\,\text{J}.$$

It can be easily noted that the same result is obtained by simply multiplying 2π by the average torque.

(d)

To obtain the energy dissipated in the loop over one turn, we calculate the power dissipated by the Joule effect:

$$E_{\text{diss}} = \int_0^T I^2(t) R \, dt = \frac{B^4 l^8 \omega^2}{R^2} \int_0^T \sin^2 \omega t \, dt = \frac{\pi B^2 l^4 \omega}{R} = 1.05 \times 10^{-3}\,\text{J}.$$

As expected, the same result as obtained above corresponds to the work done by the external mechanical torque. All the work done by the torque is dissipated due to the Joule effect.